Greek Maxima[1]

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When managing the risk of options it is often useful to know how sensitivities will change over time and with the price of the underlying. For example, many people know that gamma tends to be highest when the underlying price is close to the strike price, when the option is close to being at-the-money. Is gamma highest exactly at-the-money, or just close to at-the-money, though? The answer is that gamma is highest close to at-the-money. To get this answer we can take the derivative of gamma with respect to the underlying price, and find where the derivative is zero. Calculating these derivatives is straightforward, but often tedious. It is easy to make mistakes.

In this paper we find the extremum of gamma, and theta for European options on non-dividend paying stocks. If you are more interested in the final results than in the derivation, just skip to the summary at the end.

Preliminaries

We start with the Black Scholes formula for the price of a European call option on a non-dividend paying stock:

$$c = S\Phi(d_1) - Xe^{-rT}\Phi(d_2)$$

Equation 1

Here, S is the price of the underlying security, X is the strike price, r is the risk free rate, and T is the time to expiry. Further, we have used Φ is used to indicate the standard normal cumulative distribution function (cdf). The variables d_1 and d_2 are defined as:

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Equation 2

Here σ is the implied volatility of the underlying. The first derivative of the call price with respect to the underling is the delta of the option:

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$$\Delta = \frac{dc}{dS}$$

Equation 3

The formula for delta is well known, but we will need to use some of the same tricks later on that we need to derive the formula for delta. For this reason, and as a warm up, we first find the explicit formula for delta. To solve for delta we start by inserting Equation 1 into Equation 3:

$$\Delta = \frac{d}{dS}\left[S\Phi(d_1) - Xe^{-rT}\Phi(d_2)\right]$$
$$\Delta = S\frac{d\Phi(d_1)}{dS} + \frac{dS}{dS}\Phi(d_1) - Xe^{-rT}\frac{d\Phi(d_2)}{dS}$$
$$\Delta = S\frac{d\Phi(d_1)}{dS} + \Phi(d_1) - Xe^{-rT}\frac{d\Phi(d_2)}{dS}$$

Equation 4

The only tricky part, so far, is that we needed to apply the product rule for derivatives to the first term in Equation 1. We have to do this because the term $\Phi(d_1)$ is a function of S (S appears in the formula for d_1). In order to proceed further we need to be able to solve for the derivative of $\Phi(d_1)$ and $\Phi(d_2)$ with respect to S. There is no explicit formula for the standard normal cdf, so this may seem like an impossible task. What we need to remember is that, for any random variable, the derivative of the cumulative distribution function is the simply the probability density function (pdf) (see Miller 2014). Using the chain rule, we have:

$$\frac{d\Phi(d_1)}{dS} = \phi(d_1)\frac{dd_1}{dS} = \phi(d_1)\frac{1}{S\sigma\sqrt{T}}$$

Equation 5

Here ϕ is used to denote the standard normal pdf. For a variable x, the standard normal pdf is:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Equation 6

The standard normal pdf of d_1 is then:

$$\phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$

Equation 7

The formula for the pdf of d_2 is similar. Substituting Equation 5 into Equation 4, we have:

$$\Delta = S\phi(d_1)\frac{1}{S\sigma\sqrt{T}} + \Phi(d_1) - Xe^{-rT}\phi(d_2)\frac{1}{S\sigma\sqrt{T}}$$

Equation 8

At this point, we will find it useful to express the pdf of d_2 in terms of d_1, first we have:

$$(d_2)^1 = \left(d_1 - \sigma\sqrt{T}\right)^2 = d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T$$
$$(d_2)^1 = d_1^2 - 2\left[\ln\left(\frac{S}{X}\right) + rT\right]$$
$$-\frac{1}{2}d_2^2 = -\frac{1}{2}d_1^2 + \ln\left(\frac{S}{X}\right) + rT$$

Therefore

$$\phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$$
$$\phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}e^{\ln\left(\frac{S}{X}\right)}e^{rT}$$
$$\phi(d_2) = e^{rT}\frac{S}{X}\phi(d_1)$$

Equation 9

Substituting into Equation 8, we have:

$$\Delta = S\phi(d_1)\frac{1}{S\sigma\sqrt{T}} + \Phi(d_1) - S\phi(d_1)\frac{1}{S\sigma\sqrt{T}}$$

The first and last terms cancel out, leaving us with:

$$\Delta = \Phi(d_1)$$

Equation 10

Equation 10 is the final formula for delta. We can go no further.

Next, we need to find the derivative of delta with respect to *S*. This is just the second derivative of the price with respect to *S*, which we refer to as gamma:

$$\Gamma = \frac{d^2c}{dS^2} = \frac{d\Delta}{dS}$$

Equation 11

Substituting our formula for delta, Equation 10, we have:

$$\Gamma = \frac{d\Phi(d_1)}{dS}$$

Equation 12

We found this result earlier in Equation 5. The final formula for gamma is then:

$$\Gamma = \frac{\phi(d_1)}{S\sigma\sqrt{T}}$$

Equation 13

Gamma Maximum

Now we are ready to answer our first question: where is gamma highest. To do this, we first find the derivative of gamma with respect to *S*. This is sometimes referred to as "speed":

$$
\begin{aligned}
\frac{d\Gamma}{dS} &= \frac{1}{\sigma\sqrt{T}}\frac{d}{dS}\frac{\phi(d_1)}{S} \\
\frac{d\Gamma}{dS} &= \frac{1}{\sigma\sqrt{T}}\left[\frac{1}{S}\frac{d\phi(d_1)}{dS} - \frac{\phi(d_1)}{S^2}\frac{dS}{dS}\right] \\
\frac{d\Gamma}{dS} &= \frac{1}{\sigma\sqrt{T}}\frac{1}{S^2}\left[S\frac{d\phi(d_1)}{dS} - \phi(d_1)\right]
\end{aligned}
$$

Equation 14

To go any further, we first need to find the derivative of the standard normal pdf of d_1 with respect to S:

$$
\begin{aligned}
\frac{d\phi(d_1)}{dS} &= \frac{d}{dS}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} \\
\frac{d\phi(d_1)}{dS} &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\,(-d_1)\frac{dd_1}{dS} \\
\frac{d\phi(d_1)}{dS} &= -d_1\phi(d_1)\frac{dd_1}{dS} \\
\frac{d\phi(d_1)}{dS} &= -d_1\phi(d_1)\frac{1}{S\sigma\sqrt{T}}
\end{aligned}
$$

Equation 15

Substituting back into Equation 14, we have:

$$
\begin{aligned}
\frac{d\Gamma}{dS} &= \frac{1}{\sigma\sqrt{T}}\frac{1}{S^2}\left[-d_1\phi(d_1)\frac{1}{\sigma\sqrt{T}} - \phi(d_1)\right] \\
\frac{d\Gamma}{dS} &= -\frac{1}{\sigma\sqrt{T}}\frac{1}{S^2}\phi(d_1)\left[\frac{d_1}{\sigma\sqrt{T}} + 1\right] \\
\frac{d\Gamma}{dS} &= -\frac{\Gamma}{S}\left[\frac{d_1}{\sigma\sqrt{T}} + 1\right]
\end{aligned}
$$

Equation 16

To find the maximum, we need to find S^*, such that this derivative is zero:

$$
\begin{aligned}
-\frac{\Gamma}{S^*}\left[\frac{d_1}{\sigma\sqrt{T}} + 1\right] &= 0 \\
\frac{d_1}{\sigma\sqrt{T}} &= -1 \\
d_1 &= -\sigma\sqrt{T}
\end{aligned}
$$

Equation 17

This intermediate result will be useful further on. Continuing to solve for S^*:

$$\frac{\ln\left(\frac{S^*}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=-\sigma\sqrt{T}$$
$$\ln\left(\frac{S^*}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)T=-\sigma^2T$$
$$\ln\left(\frac{S^*}{X}\right)=-\left(r+\frac{3\sigma^2}{2}\right)T$$
$$S^*=Xe^{-\left(r+\frac{3\sigma^2}{2}\right)T}$$

Equation 18

Near expiry, when *T* is small, the right-hand side of Equation 18 is very close to *X*. Many practitioners remember that near expiry, gamma is highest at-the-money. As Equation 18 makes clear, far from expiry, or if *r* or *σ* are high, the maximum could be significantly lower than *X*. Though this derivation started with the formula for the price of a call, the result is also true for puts.

To be certain that this is the maximum and not the minimum, we should show that the derivative of Equation 16 is negative. It is not too difficult to show that:

$$\frac{d^2\Gamma}{dS^2}=\frac{\Gamma}{S^2\sigma^2T}\left[d_1^2+3d_1\sigma\sqrt{T}+2\sigma^2T+1\right]$$

Equation 19

At S^*, substituting in Equation 17, this simplifies to:

$$\frac{d^2\Gamma}{dS^2}=-\frac{\Gamma}{S^{*2}\sigma^2T}$$

Equation 20

Equation 20 is always negative, so S^* is indeed a maximum.

Theta Maximum

Theta is the derivative of an options price with respect to time:

$$\Theta=\frac{dc}{dt}$$

Equation 21

First a note on the sign of theta. Theta is meant to show how the price of an option changes over time. In Equation 1, however, we have defined the price in terms of time to expiry, *T*, not time, *t*. While not necessary, working with *T* rather than *t*, results in more compact equations. The only problem is that time to expiry gets shorter over time. Time to expiry moves in the opposite direction as time. So in terms of *T*, we have:

$$\Theta = -\frac{dc}{dT}$$

Equation 22

Theta tends to be negative for both puts and calls. That is, as time goes by, long option positions tend to become less valuable. We often refer to this loss in value as time decay. This is only a rule of thumb, and for deep in the money puts, theta can be positive.

For a European call option on a non-dividend paying stock, the Black Scholes theta is:

$$\Theta = -\frac{\sigma}{2\sqrt{T}} S\phi(d_1) - rXe^{-rT}\Phi(d_2)$$

Equation 23

Theta is clearly a function of the underlying price, *S*, and we would like to know at which price theta is the greatest, that is at what underlying price is theta the most negative. To do this, we need to calculate the derivative of theta with respect to S, which is often referred to as “charm”:

$$\text{charm} = \frac{d\Theta}{dS}$$

Equation 24

We proceed as follows:

$$\frac{d\Theta}{dS} = -\frac{\sigma}{2\sqrt{T}}\left[\phi(d_1) + S\frac{d\phi(d_1)}{dS}\right] - rXe^{-rT}\phi(d_2)\frac{dd_2}{dS}$$

Equation 25

Using earlier results, Equation 15 and Equation 9, we have:

$$\frac{d\Theta}{dS} = -\frac{\sigma}{2\sqrt{T}}\left[\phi(d_1) - d_1\phi(d_1)\frac{1}{\sigma\sqrt{T}}\right] - rXe^{-rT}e^{rT}\frac{S}{X}\phi(d_1)\frac{1}{S\sigma\sqrt{T}}$$

$$\frac{d\Theta}{dS} = \frac{\sigma}{2\sqrt{T}}\left[d_1\phi(d_1)\frac{1}{\sigma\sqrt{T}} - \phi(d_1)\right] - \frac{r}{\sigma\sqrt{T}}\phi(d_1)$$
$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{\sqrt{T}}\frac{\sigma}{2}\left[d_1\frac{1}{\sigma\sqrt{T}} - 1\right] - \frac{\phi(d_1)}{\sqrt{T}}\frac{r}{\sigma}$$
$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{\sqrt{T}}\frac{1}{2}\left[\frac{d_1}{\sqrt{T}} - \sigma\right] - \frac{\phi(d_1)}{\sqrt{T}}\frac{r}{\sigma}$$
$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{\sqrt{T}}\left[\frac{d_1}{2\sqrt{T}} - \frac{\sigma}{2} - \frac{r}{\sigma}\right]$$

We can rearrange this further:

$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{\sqrt{T}}\frac{1}{2\sigma}\frac{1}{T}\left[\sigma\sqrt{T}d_1 - 2\left(r + \frac{\sigma^2}{2}\right)T\right]$$

$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{T^{3/2}}\frac{1}{2\sigma}\left[\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T - 2\left(r + \frac{\sigma^2}{2}\right)T\right]$$
$$\frac{d\Theta}{dS} = \frac{\phi(d_1)}{T^{3/2}}\frac{1}{2\sigma}\left[\ln\left(\frac{S}{X}\right) - (r + \frac{\sigma^2}{2})T\right]$$

Equation 26

To find where theta is maximum, we need to find the value of S, S^*, that sets Equation 26 equal to zero. The standard normal pdf is always positive, so even though *S* is in d_1, $\phi(d_1)$ can never be zero. To set Equation 26 equal to zero, then, we need to find:

$$\ln\left(\frac{S^*}{X}\right) - \left(r + \frac{\sigma^2}{2}\right)T = 0$$

Equation 27

Solving, we have:

$$S^* = Xe^{\left(r+\frac{\sigma^2}{2}\right)T}$$

Equation 28

In order to prove that S^* is a minimum and not a maximum, we should show that the second derivative with respect to *S* is positive. It takes some work, but eventually we would find:

$$\frac{d^2\Theta}{dS^2} = -\frac{\phi(d_1)}{T^{3/2}}\frac{1}{2\sigma}\frac{1}{S}\frac{1}{\sigma^2 T}\left[\left(\ln\left(\frac{S}{X}\right) - (r + \frac{\sigma^2}{2})T\right)\left(\ln\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2})T\right) - \sigma^2 T\right]$$

Equation 29

This looks a bit unwieldy, but if we substitute our condition from Equation 27 into the equation, we have:

$$\frac{d^2\Theta}{dS^2} = \frac{\phi(d_1)}{T^{3/2}}\frac{1}{2\sigma}\frac{1}{S^*}$$

Equation 30

All of these terms must be positive, so the second derivative is positive and S^* is indeed a minimum. Though we did not prove it here, this result is also true for European puts on non-dividend paying stocks.

Summary

For both gamma and theta, the extremum occur close to at-the-money near expiration.

The maximum for gamma for both European calls and puts on non-dividend paying stocks occurs at:

$$S^* = Xe^{-\left(r+\frac{3\sigma^2}{2}\right)T}$$

For theta, the minimum, the most negative value, occurs at:

$$S^* = Xe^{\left(r+\frac{\sigma^2}{2}\right)T}$$

References

Miller, Michael B. 2014. *Mathematics and Statistics for Financial Risk Management*. Hoboken, NJ: John Wiley & Sons, Inc.